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Hamiltonian Systems with Constraints: A Geometric Approach

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Hamilton–Dirac equations for a constrained Hamiltonian system are deduced from a variational principle. In the local problem for such systems an algorithm is proposed to obtain the final constraint manifold and the dynamical vector field on it using vector fields on the phase space. The global problem is solved in terms of fiber bundles associated with the problem.

1. INTRODUCTION

The geometric description of a mechanical system is formulated in two different ways, classically called Lagrangian mechanics and Hamiltonian mechanics. In the first one, the determination of the system is made by its configuration space, a differentiable manifold M, and the Lagrangian of the system, which is a real function L defined on TM, the tangent bundle of M. The Hamilton variational principle enables us to obtain the Euler–Lagrange equations and then we can calculate all the possible trajectories of the system.

In Hamiltonian mechanics, the objects are the configuration space M and the Hamiltonian of the system H, which is a real function defined on the phase space T^*M, the cotangent bundle of M. By means of the canonical 2-form on T^*M we obtain the Hamiltonian vector field and by integration the trajectories of the system.

The relation between both formulations is the Legendre transformation, $FL: TM \to T^*M$. If FL is a diffeomorphism, then one formulation is the image of the other under that diffeomorphism.

Gotay *et al.* (1978; Gotay and Nester, 1979, 1980) have studied the conditions for the existence of a Hamiltonian formulation associated with a Lagrangian one. They call such Lagrangians almost regular.

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In this situation, the general problem is to consider that the image of the Legendre transformation is a proper submanifold of the cotangent bundle. We thus obtain Hamiltonian systems with constraints.

The equations of motion in this problem are the so-called Hamilton–Dirac equations and, in general, they have solution not in all T^*M, but only in a submanifold, the final constraint submanifold. This problem leads to the study of the secondary constraints. Gotay and Nester (1979) provide an algorithm to obtain the constraint submanifold where the problem can be solved, the PCA algorithm.

I propose here an alternative method to obtain the solution of this problem using tangent vector fields in T^*M which satisfy the Hamilton–Dirac equations. This algorithm allows us to obtain the constraint submanifold and the dynamical tangent vector field on that submanifold, if they exist.

I begin with the study of the local problem; that is, I suppose that the image of FL is a submanifold of T^*M defined by the zeroes of a finite family of functions. In this case the algorithm is linear. Afterwards, I solve the global problem.

This paper is organized into the following parts:

Section 2: Statement of the problem and Hamilton–Dirac equations. In this section I state the problem and set the notations. I also obtain the Hamilton–Dirac equations from a variational principle.

Section 3: The local problem. Here the image of FL is a submanifold of T^*M defined by a finite family of functions. I classify the constraints as first class and second class and develop the algorithm to obtain the constraint submanifold and the dynamical vector field on it.

Section 4: The global problem. In this section I prove the existence of an algorithm to obtain the constraint submanifold and the dynamical vector field when the image of FL is any submanifold of the phase space.

At the end of this section, I make a brief comparison with the Gotay algorithm and give an example.

I use Abraham and Marsden (1978) and de León and Rodrigues (1985) as general references. The initial ideas come from Dirac (1950, 1964). A more physical approach is in Battle *et al.* (1986). A different one is in Lichnerowicz (1975).

2. STATEMENT OF THE PROBLEM AND HAMILTON–DIRAC EQUATIONS

Let M be a differentiable manifold, $\dim M = m$, and $L: TM \to \mathbb{R}$ a C^∞ function [I write $S(M)$ for the ring of C^∞ functions on the manifold M]. Here L is called the Lagrangian of the system.

Let $FL: TM \to T^*M$ be the Legendre transformation associated with the Lagrangian L.

Following Gotay and Nester (1978), one says that L is almost regular if:

(a) FL is a submersion on a closed embedded submanifold M_0 of T^*M. One calls M_0 the manifold of the primary constraints and puts $\dim M_0 = 2m - m_0$.

(b) The fibers of the mapping FL are connected.

In this case, Gotay and Nester (1978) prove that there exists a Hamiltonian formulation associated with the given Lagrangian L. In the following I only consider almost-regular Lagrangians.

Let $A: TM \to \mathbb{R}$ be defined by $A(x, u) = (FL(x, u))(x, u)$. Here A is the action of the Lagrangian L and $E = A - L$ is the energy of the system.

Proposition (Gotay and Nester, 1978). If L is almost regular, then E is FL-projectable.

Proof. Since the fibers of FL are connected, we only have to prove that $X(E) = 0$ for any vector field X tangent to the fibers of FL.

But since FL preserves the fibers of TM, we can use a canonical system of coordinates in TM. Let (q^i, v^i) be one such system.

We know that $E = \sum v^i \, \partial L/\partial v^i - L$ and that if X is vertical in TM, then $X = \sum \lambda^j \, \partial/\partial v^j$. But if X is tangent to the fibers of FL, then its components satisfy the relation $\sum \lambda^i \, \partial^2 L/\partial v^i \, \partial v^j = 0$, $j = 1, \ldots, m$.

Now it is immediate to prove that $X(E) = 0$. ■

Corollary. There exist functions $H: T^*M \to \mathbb{R}$ with $H \circ FL = E$. ■

Note that all the functions H with this property take the same values on M_0. In the sequel I show that the solution of the problem does not depend on the chosen function H, providing that $H \circ FL = E$. Therefore, we can suppose we have chosen one and call it the Hamiltonian of the system. With this function H, the formulation of the problem is the following:

Let $\sigma: [a, b] \to M_0$ be a differentiable curve. Put $\bar{\sigma}: [a, b] \to M_0 \times \mathbb{R}$ for the curve defined by $\bar{\sigma}(t) = (\sigma(t), t)$. Let $dt \in \Omega^1(M_0 \times \mathbb{R})$ be the pullback of the standard volume element in $\mathbb{R}$.

Let $\theta \in \Omega^1(T^*M)$ be the canonical 1-form on T^*M defined by

$$\theta(U) = \gamma((T_{(p,\gamma)}\tau^*)(U)) \qquad \text{for} \quad U \in T_{(p,\gamma)}T^*M$$

where $\tau^*: T^*M \to M$ is the canonical projection. Put $\omega = -d\theta$.

For any curve $\sigma: [a, b] \to M_0$, we have the functional

$$\sigma \mapsto \int_{\bar{\sigma}} \theta - H \, dt$$

Definition. The varional problem associated with this functional for the curves σ such that $\sigma(a)=A$ and $\sigma(b)=B$, A and B fixed points in T^*M, is called Hamilton–Dirac variational problem.

According to Sternberg (1964) if σ is a solution of this problem then

$$\int_{\tilde{\sigma}} L_D(\theta - H\,dt) = 0$$

for every vector field $D \in \mathscr{X}(M_0)$ with $D(A)=0$, $D(B)=0$, where L_D is the Lie derivative with respect to the vector field D.

Proposition. If σ is a solution of the Hamilton–Dirac variational problem and σ' is the tangent vector to σ then:

$$i^*(i_{\sigma'}\,d\theta - dH) = 0$$

where i: $M_0 \to T^*M$ is the natural injection.

Proof. We have:

$$0 = \int_{\tilde{\sigma}} L_D(\theta - H\,dt) = \int_{\tilde{\sigma}} i_D(d(\theta - H\,dt)) + \int_{\tilde{\sigma}} d(i_D(\theta - H\,dt))$$

but:

$$\begin{aligned}\int_{\tilde{\sigma}} d(i_D(\theta - H\,dt)) &= \int_a^b \tilde{\sigma}^*(d(i_D(\theta - H\,dt))) \\ &= \int_a^b d(\tilde{\sigma}^*(i_D(\theta - H\,dt)) = 0\end{aligned}$$

because $D(a)=0$, $D(b)=0$.

Then

$$\begin{aligned}0 = \int_{\tilde{\sigma}} i_D(d(\theta - H\,dt)) &= \int_a^b \tilde{\sigma}^*(i_D(d(\theta - H\,dt))) \\ &= \int_a^b (d\theta - dH \wedge dt)(D, \tilde{\sigma}')\,dt \\ &= -\int_a^b i_D(i_{\tilde{\sigma}'}(d\theta - dH \wedge dt))\,dt\end{aligned}$$

but D is any vector field tangent to M_0 which verifies $D(A)=0$, $D(B)=0$, so that:

$$i^*(i_{\tilde{\sigma}'}(d\theta - dH \wedge dt)) = 0$$

Notice that

$$i_{\tilde{\sigma}'}(d\theta - dH \wedge dt) = i_{\tilde{\sigma}'}\,d\theta - \tilde{\sigma}'(H)\,dt + dH$$

Then, using that $i_D(\bar{\sigma}'(H)\,dt)=0$ and $\omega=-d\theta$ is the canonical 2-form on T^*M, we have

$$i^*(i_{\bar{\sigma}'}\omega - dH)=0$$

or equivalently

$$i^*(i_{\sigma'}\omega - dH)=0 \quad \blacksquare$$

2.1. Hamilton–Dirac Equations

According to the above proposition, if Z is a tangent vector field to T^*M whose integral curves are solutions of the Hamilton–Dirac problem, then Z has to satisfy the following conditions:

1. $i^*(i_Z\omega - dH)=0$.
2. Z is tangent to M_0.

These are the Hamilton–Dirac equations for Z.

Remark. If we change H by $\tilde{H}$ with the condition $FL^*H=FL^*\tilde{H}$, this does not modify the solution of the problem, because $i^*H=i^*\tilde{H}$, and then the vector field Z is the same for H or $\tilde{H}$.

Notice that we can now state the problem independently of its Lagrangian origin, because only the manifold M, the submanifold M_0 of T^*M, and the Hamiltonian H are used.

3. THE LOCAL PROBLEM

In this section, I assume the following hypothesis:

Hypothesis 1. The submanifold M_0 is the set of zeros of a finite family of functions $\phi_1, \phi_2, \ldots, \phi_{m_0}$, with $\dim M_0 = 2m - m_0$. The functions ϕ_i are called primary constraints.

Lemma. Let N be a differentiable manifold, $\dim N = n$, and T be the submanifold defined by $\phi_1=\phi_2=\cdots=\phi_h=0$, with $\dim T = n-h=k$, and $\phi_i \in S(N)$. Let $i\colon T\to N$ be the natural injection.

If $\omega \in \Omega^1(N)$ satisfies $i^*\omega=0$, then there exist functions f^i in $S(N)$ with $\omega(x)=\sum f^i(x)\,d\phi_i(x)$ for all x in T.

Proof. For a point p in T, there exists an open neighborhood U in N, $p\in U$, and functions $\psi_1, \psi_2, \ldots, \psi_k$ such that $\phi_1, \ldots, \phi_h, \psi_1, \ldots, \psi_k$ form a local system in U. Then

$$\omega|_U = \sum f^i\,d\phi_i + \sum g^j\,d\psi_j$$

for some f^i and g^j in $S(N)$.

The condition $i^*\omega=0$ implies that $g^j(x)=0$ for all $x\in U\cap T$. Then $\omega(x)=\sum f^i(x)\,d\phi_i(x)$ for all $x\in U\cap T$.

If p and q are two different points in T, U and V the corresponding neighborhoods, and f^i and $\tilde{f}^i$ the associated functions, then $f^i(y) = \tilde{f}^i(y)$ for all $y \in U \cap V$ because $d\phi_1, \ldots, d\phi_h$ are independent. So the functions f^i are globally defined on T. We can extend them to all N because T is a closed submanifold. ■.

According to this lemma, the Hamilton–Dirac problem can be stated as follows.

Is there any vector field $Z \in \mathscr{X}(T^*M)$ such that:

1. Z is tangent to M_0.
2. $(i_Z\omega - dH)(x) = \sum f^i(x)\, d\phi_i(x)$ if $x \in M_0$?

This is the standard form of the Hamilton–Dirac equations.

Observe that the form of this equations does not change if we choose another system of functions $\bar{\phi}_1, \ldots, \bar{\phi}_{m_0}$ which define the same submanifold M_0.

3.1. Geometric Interpretation of the Equations

Consider the following vector fields on T^*M:

$$X \in \mathscr{X}(T^*M) \qquad \text{such that} \quad i_X\omega = dH$$

$$X_i \in \mathscr{X}(T^*M) \qquad \text{such that} \quad i_{X_i}\omega = d\phi_i, \; i = 1, \ldots, m_0$$

These last vector fields are linearly independent at every point because so are $d\phi_i$.

With these vector fields we can enunciate the Hamilton–Dirac problem as follows.

Are there functions f^i in $S(T^*M)$, $i = 1, \ldots, m_0$, such that the vector field

$$Z = X + \sum f^i X_i$$

is tangent to M_0?

The condition of being tangent to M_0 is $(Z\phi_i)(x) = 0$ for all $x \in M_0$. We use the notation $Z\phi_i \underset{\circ}{=} 0$ for that condition.

Observe that if one X_i is tangent to M_0, then the corresponding coefficient f^i can be an arbitrary function. Then we can classify the constraints into two different types.

3.2. Classification of Constraints

Definitions:

Let Ω be the restriction of ω to the submodule generated by the vector fields $X_1, \ldots, X_{m_0}$.

For $x \in M_0$, we say that ϕ_i is a first-class constraint in x if $X_i(x)$ is tangent to M_0, that is, $(X_i\phi_j)(x) = 0$, for $j = 1, \ldots, m_0$.

In any other case we say that ϕ_i is a second-class constraint.

Proposition. ϕ_i is a first-class constraint at $x \in M_0$ if and only if $X_i(x)$ is in the radical of $\Omega(x)$.

Proof. The tangent vector $X_i(x)$ is in the radical of $\Omega(x)$ if $\omega(X_i, X_j)(x) = 0$ for $j = 1, \ldots, m_0$, that is, whenever $(X_i\phi_j)(x) = 0$ for $j = 1, \ldots, m_0$. ∎

The following hypothesis is assumed in the rest of this section.

Hypothesis 2. (a) The dimension of the radical of $\Omega(x)$ is independent of the point $x \in M_0$. Let $m_1 \leq m_0$ be such dimension.

(b) There exist functions $f_i^j \in S(T^*M)$, $j = 1, \ldots, m_0$, $i = 1, \ldots, m_1$ such that the vector fields

$$Y_i = \sum f_i^j X_j, \qquad i = 1, \ldots, m_1$$

form a basis of the radical of $\Omega(x)$ for all $x \in M_0$, and there exists a minor of the matrix (f_i^j) whose determinant is different from zero at any point $x \in M_0$.

Remark. Notice that locally (a) implies (b).

Definitions:

Let $\phi_1^0, \ldots, \phi_{m_1}^0$ be functions defined by $\phi_i^0 = \sum f_i^j \phi_j$, where f_i^j are the functions of hypotheses 2b. Then $d\phi_1^0, \ldots, d\phi_{m_1}^0$ are independent at any point of M_0 because so are $Y_1, \ldots, Y_{m_1}$. So we can choose $\phi_{m_1+1}^0, \ldots, \phi_{m_0}^0$ among $\phi_1, \ldots, \phi_{m_0}$ such that $\phi_1^0, \ldots, \phi_{m_0}^0$ define the submanifold M_0.

We say that $\phi_1^0, \ldots, \phi_{m_1}^0$ are first-class constraints and that $\phi_{m_1+1}^0, \ldots, \phi_{m_0}^0$ are second-class constraints.

The functions $\phi_1^0, \ldots, \phi_{m_0}^0$ are called primary constraints.

Proposition. Let X_i^0 be vector fields on T^*M such that $(i_{X_i^0})\omega = d\phi_i^0$ for $i = 1, \ldots, m_0$. Then:

(a) If $x \in M_0$, then $X_1^0(x), \ldots, X_{m_1}^0(x)$ generate the radical of $\Omega(x)$.

(b) $X_1^0, \ldots, X_{m_1}^0$ are tangent to M_0.

Proof. (a) We have $d\phi_i^0 = \sum f_i^j \, d\phi_j + \sum \phi_j \, df_i^j$ for $i = 1, \ldots, m_0$; then, if x is a point in M_0,

$$d\phi_i^0(x) = \sum f_i^j(x) \, d\phi_j(x)$$

Then $X_i^0(x) = Y_i(x)$, $i = 1, \ldots, m_1$, and the result follows.

(b) Direct consequence of (a). ∎

The terminology of first-class constraints for the functions ϕ_i^0 is not justified: They are first class at all points of M_0.

3.3. Algorithm to Reduce the Dimension

With the new set of primary constraints ϕ_i^0, $i=1,\ldots,m_0$, we state the problem as follows.

Are there functions $f^i \in S(T^*M)$ such that the vector field

$$Z = X + \sum f^i X_i^0$$

is tangent to M_0?

Then the system to solve is

$$X\phi_j^0 + \sum f^i X_i^0 \phi_j^0 \underset{\circ}{=} 0, \qquad j=1,\ldots,m_0$$

That is,

$$X\phi_j^0 \underset{\circ}{=} 0, \qquad j=1,\ldots,m_1$$

$$\sum f^i \omega(X_i^0, X_j^0) \underset{\circ}{=} -X\phi_j^0, \qquad j>m_1$$

This system has a solution only at the points P of M_0 such that $(X\phi_j^0)(P)=0$, $j=1,\ldots,m_1$.

Definitions:
Let M_1 be the submanifold of M_0 defined by $X\phi_j^0=0$, $j=1,\ldots,m_1$. We say that $X\phi_1^0,\ldots,X\phi_{m_1}^0$ are secondary constraints.

Remarks. Observe that $X\phi_i^0=\{H, \phi_i^0\}$. This is the form in which Dirac (1964) wrote them.

Observe that, on M_1, the system has only one solution for f^i, $i>m_1$, because the matrix $\omega(X_i^0, X_j^0)$, $i, j>m_1$, has rank m_0-m_1.

Now, the situation is the same as at the beginning, that is, the Hamilton-Dirac problem, but on the manifold M_1 given by the zeroes of the functions $\phi_1^0,\ldots,\phi_{m_0}^0$, $X\phi_1^0,\ldots,X\phi_{m_1}^0$. We must find an independent family, if it exists, in order to be in the same situation as at the beginning of this section.

Then, following this method, we have a chain of submanifolds

$$T^*M \supset M_0 \supset M_1 \supset \cdots$$

Any one of them is given by the zeroes of a finite family of functions and so we can iterate the process.

This process is necessarily finite, because M is a finite-dimensional manifold. Then the only possible situations are:

1. There exists an integer $r>0$ such that $M_r=\varnothing$.
2. There exists an integer $r>0$ such that M_r is not empty, but $\dim M_r=0$.
3. There exists an integer $r>0$ such that $M_r=M_{r+h}$ for all $h>0$ and $\dim M_r>0$.

In the first case the problem has no solution. It is inconsistent, according to Dirac (1964). In the second one there is no dynamics, and in the third one we obtain the final constraint manifold.

Observe that we have used two different hypotheses in this study of the local problem. We leave out the first one when we study the global problem. But keep the hypothesis 2a, constancy of the dimension of the radical of the 2-form Ω. But remember that part b of hypothesis 2 holds true locally when we assume part a.

4. THE GLOBAL PROBLEM

Let M_0 be a submanifold of T^*M, dim $M_0 = n = 2m - m_0$. For M_0 we state the Hamilton-Dirac problem as follows.

Is there any vector field Z on T^*M tangent to M_0 which satisfies

$$i^*(i_Z\omega - dH) = 0$$

where $i\colon M_0 \to T^*M$ is the natural injection?

Definitions. Consider the following sets:

$$I_0 = \{f \in S(T^*M);\ i^*f = 0\}$$

$$dI_0 = \{df;\ f \in I_0\}$$

$\{dI_0\}$ = the submodule of $\Omega^1(T^*M)$ generated by dI_0, that is, $\alpha \in \{dI_0\}$ if and only if $\alpha = \sum f^i\, dg_i$, with $g_i \in I_0$

$$J_0 = \{X \in \mathscr{X}(T^*M);\ i_X\omega \in \{dI_0\}\}$$

Remark. These sets have the following properties: Given a point x in M_0, there exists an open neighborhood U of x in T^*M and a family of functions $\phi_1, \ldots, \phi_m \in S(T^*M)$ such that

$$U \cap M_0 = \{x \in U;\ \phi_i(x) = 0,\ i = 1, \ldots, m_0\}$$

and, moreover, the following hold.

1. If $f \in I_0$ then

$$f|_U = \sum_{i=1}^{m_0} g^i\phi_i|_U$$

for some $g^i \in S(T^*M)$.

2. From condition 1 we deduce that if $\alpha \in dI_0$, then

$$\alpha|_U = \sum_i g^i\, d\phi_i|_U + \sum_i \phi_i\, dg^i|_U$$

for some $g^i \in S(T^*M)$.

Observe that

$$\alpha|_{U\cap M_0} = \sum g^i \, d\phi_i|_{U\cap M_0}$$

3. For $\{dI_0\}$, the above conditions imply the following: If $\beta \in \{dI_0\}$, then

$$\beta = \sum_i f^i \, dg_i \qquad \text{with} \quad g_i \in I_0$$

Thus

$$g_i|_U = \sum_j h_i^j \phi_j|_U$$

and therefore

$$\beta|_U = \sum_{i,j} f^i h_i^j \, d\phi_j|_U + \sum_{i,j} f^i \phi_j \, dh_i^j|_U$$

Hence

$$\beta|_{U\cap M_0} = \sum_{i,j} f^i h_i^j \, d\phi_j|_{U\cap M_0} = \sum_j k^j \, d\phi_j|_{U\cap M_0}$$

where $k^j \in S(T^*M)$.

4. If $X \in J_0$, then $i_X\omega \in \{dI_0\}$, hence in U we have $i_X\omega|_U = \beta|_U$, that is,

$$i_X\omega|_{U\cap M_0} = \sum_{i,j} f^i h_i^j \, d\phi_j|_{U\cap M_0} = \sum_j k^j \, d\phi_j|_{U\cap M_0}$$

So, if we take the vector fields $X_j \in \mathscr{X}(T^*M)$ with $i_X\omega = d\phi_j$, we have

$$X|_{U\cap M_0} = \sum_{i,j} f^i h_i^j X_j|_{U\cap M_0} = \sum_{j'} k^j X_j|_{U\cap M_0}$$

for some $k^j \in S(T^*M)$.

4.1. Bundles Associated with the Problem

One has that $J_0(M_0) = \bigcup_{x\in M_0} J_0(x)$ with the natural projection on M_0 is a vector bundle with rank m_0 on the manifold M_0. The $J_0(x)$ are the values of all vector fields in J_0 at the point $x \in M_0$.

Let Ω be the restriction of ω to J_0 and $R_0(x) = \operatorname{rad}\Omega(x)$ for any $x \in M_0$.

Hypothesis. The subspace $\operatorname{rad}\Omega(x)$ has the same dimension at every point $x \in M_0$.

Let m_1 be such dimension. We have $m_1 \le m_0$.

Let $R_0(M_0) = \bigcup_{x\in M_0} R_0(x)$, then $R_0(M_0)$ with the natural projection on M_0 is a vector subbundle of $J_0(M_0)$.

Observe that the sections of $J_0(M_0)$ are the vector fields of J_0 restricted to M_0.

Definition. Let $X \in \mathscr{X}(T^*M)$ be the only vector field such that $i_X\omega = dH$ and let

$$M_1 = \{x \in M_0;\ \omega(X, Y)(x) = 0,\ Y \in \operatorname{Secc} R_0(M_0)\}$$
$$= \{x \in M_0;\ Y_x(H) = 0,\ Y \in \operatorname{Secc} R_0(M_0)\}$$

M_1 is a submanifold of M_0 and we assume that M_1 is not empty.

Theorem. There exists a section Y of the bundle $J_0(M_0)$ such that:

1. The vector field $Z = X + Y$ is tangent to M_0 at the points of M_1.
2. $i^*(i_Z\omega - dH) = 0$
   [that is, if $x \in M_0$ and $v \in T_xM_0$, then $(i_Z\omega - dH)(v) = 0$].

Proof. The local result we have previously proved gives us the following corollary.

For any point $x \in M_0$ there exists an open neighborhood U of x in T^*M and a family of functions $\phi_1, \ldots, \phi_{m_0}$ such that:

1. $M_0 \cap U = \{x \in U;\ \phi_i(x) = 0,\ i = 1, \ldots, m_0\}$.

2. If $X_i \in \mathscr{X}(T^*M)$ satisfy $i_{X_i}\omega = d\phi_i$, then $\omega(X_i, X_j)(x) = 0$ for $x \in M_0 \cap U$, $i \leq m_1$, $j = 1, \ldots, m_0$, that is, the vector fields $X_1, \ldots, X_{m_1}$ generate the radical of Ω, restriction of ω to $X_1, \ldots, X_{m_0}$, at the points of $M_0 \cap U$.

3. Let $M_1(U) = \{x \in M_0 \cap U;\ \omega(X, X_j) = 0,\ j = 1, \ldots, m_1\}$, then there exists one and only one vector field $Y \in \mathscr{X}(T^*M)$ with $Y = \sum_{i > m_1} f^i X_i$ such that the vector field $Z = X + Y$ satisfies (a) Z is tangent to M_0 at the points of $M_1(U)$, and (b) $i^*(i_Z\omega - dH) = 0$.

Observe that $i_Y\omega = \sum_{i > m_1} f^i\, d\phi_i$.

Now it suffices to construct a global section. For this let $\{U_i;\ i \in I\}$ be a family of open neighborhoods with the following properties:

1. They are solutions of the local problem.
2. The family $U_i \cap M_0$, $i \in I$, is a locally finite open covering of M_0.

Let $\{\eta_i;\ i \in I\}$ be a partition of unity with functions of $S(M_0)$ subordinated to the covering. If Y_i, $i \in I$, are the solutions of the local problems for the open sets U_i, then $Y = \sum \eta_i Y_i$ is a section of the bundle $J_0(M_0)$.

Consider, at the points of M_0, the vector field $Z = X + Y$. We have:

1. Z is tangent to M_0 at the points of M_1. Indeed, if $x \in M_1$ and $f \in I_0$, then

$$Z(f)(x) = (X + \textstyle\sum \eta_i Y_i)(f)(x) = \sum \eta_i(x)((X + Y_i)_x(f)) = 0$$

because $M_1 \cap U_i = M_1(U_i)$.

2. $i^*(i_Z\omega - dH) = 0$, because if $x \in M_0$ and $v \in T_xM_0$, then

$$\omega(Z, v) - v(H) = \textstyle\sum \eta_i(x)\omega(Y_i, v) = 0$$

Hence the result follows. ■

Remarks.

1. Is there any relation between two different solutions of the problem? Suppose Y and $\tilde{Y}$ are solutions, that is, the vector fields

$$Z = X + Y, \qquad \tilde{Z} = X + \tilde{Y}$$

satisfy the conditions of the theorem. Note that $Z - \tilde{Z}$ is tangent to M_0 and satisfies $i^*(i_{(Z-\tilde{Z})}\omega) = 0$. Then $Z - \tilde{Z}$ is a solution of the homogeneous problem, that is, the problems in which $i^*H = 0$. The same is true for $Y - \tilde{Y}$.

2. The vector fields Y_i, solutions of the local problems, are defined at every point in T^*M, not only in M_0. If M_0 is closed, then we can extend a solution Y defined in M_0 to all T^*M by extending the functions η_i.

4.2. Conclusion

As in the local case, we obtain a chain of submanifolds

$$M_0 \supset M_1 \supset M_2 \supset \ldots$$

and, since the situation is the same as in the local problem, we obtain the same conclusions.

4.3. Relation with PCA Algorithm

Gotay and Nester (1978) make use of the presymplectic structure defined on M_0 by the restriction of ω. In this paper I use only the symplectic structure of the phase space T^*Q. This structure is easier to use, from the point of view of calculus. In addition, the calculus of the secondary constraints is made directly.

I prove that the local structure of the algorithm is the same as in the global situation. The global version is made directly from the local one.

Finally, the algorithm allows one to calculate the "final" constraint submanifold and at the same time the dynamical vector field on this submanifold.

On the other hand, the PCA algorithm can be made in a presymplectic manifold which is not immersed in a symplectic one.

4.4. An Example

Take the Lagrangian $L = v_1^2 + q_3 v_4 + q_2^2 + q_4^2$ on $T\mathbb{R}^4$. We obtain

$$M_0 = \{(q, p);\ \phi_1 = p_2 = 0,\ \phi_2 = p_3 = 0,\ \phi_3 = p_4 - q_3 = 0\} \subset T^*\mathbb{R}^4$$

$$H = \tfrac{1}{4}p_1^2 - q_2^2 - q_4^2$$

The Hamiltonian vector field is $X=\frac{1}{2}p_1\,\partial/\partial q_1+2q_2\,\partial/\partial p_2+2q_4\,\partial/\partial p_4$ and we obtain from the constraints

$$X_1=\partial/\partial q_2, \qquad X_2=\partial/\partial q_3, \qquad X_3=\partial/\partial q_4+\partial/\partial p_3$$

Then ϕ_1 is a first-class and ϕ_2 and ϕ_3 are second-class constraints. Hence the only secondary constraints is $X\phi_1=0$, that is, $q_2=0$.

Then we have

$$M_1=\{(q, p);\ p_2=0,\ p_3=0,\ p_4-q_3=0,\ q_2=0\}$$

as the new constraint submanifold.

On M_1, the vector fields X_i associated with the constraints are second class, hence there is no other constraint.

The dynamical vector field on M_1 is

$$Z=\tfrac{1}{2}p_1\,\partial/\partial q_1+2q_4\,\partial/\partial q_3+2q_4\,\partial/dp_4$$

that is, Z is tangent to M_1 and $Z=X+\sum\lambda^iX_i$.

This is an example in which the present algorithm is easier to calculate than the PCA algorithm.

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